9.10: Taylor and Maclaurin Series

When you finish your homework you should be able to…

- π Find a Taylor series or a Maclaurin series for a function.
- π Find a binomial series.
- π Use a basic list of Taylor series to derive other power series.

WARM-UP: Find the 8th degree Maclaurin polynomial for the function $f(x) = \cos x$. $= f^{(4)}(x)$

$f'(x) = -\sin x = f^{(5)}(x)$

$f''(x) = -\cos x = f^{(6)}(x)$

$f'''(x) = \sin x = f^{(7)}(x)$

$f^{(8)}(x) = \cos x$

$f(0) = f^{(4)}(0) = f^{(8)}(0) = 1$

$f'(0) = f^{(5)}(0) = 0$

$f''(0) = f^{(6)}(0) = -1$

$f'''(0) = f^{(7)}(0) = 0$

$$P_8(x) = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \frac{0x^7}{7!} + \frac{1x^8}{8!}$$

$$\boxed{P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}}$$

Now let's see if we can form a power series!

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad (-\infty, \infty)$$

What about that interval of convergence?

Ratio test:

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2 \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \right| = 0 < 1$$

IOC $(-\infty, \infty)$

Theorem: The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all x in an open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Pf: Consider $\sum a_n (x-c)^n$ with a radius of convergence R.
We know that the nth derivative of f exists for $|x-c| < R$.
[props of functions defined by power series theorem]

$$f^{(0)}(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdots$$

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots$$

$$f''(x) = 2a_2 + 3\cdot 2\cdot a_3(x-c) + 4\cdot 3\cdot a_4(x-c)^2 + \cdots$$

$$f'''(x) = 3\cdot 2\cdot a_3 + 4\cdot 3\cdot 2a_4 + \cdots$$

$$f^{(n)}(x) = n!a_n + (n+1)!a_{n+1}(x-c) + \cdots$$

And,

$$f^{(0)}(c) = 0!a_0$$

$$f'(c) = 1!a_1$$

$$f''(c) = 2!a_2$$

$$\vdots$$

$$f^{(n)}(c) = n!a_n$$

$$f^{(n)}(c) = n!a_n$$

$$a_n = \frac{f^{(n)}(c)}{n!} \quad //$$

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

is called the __Taylor__ series for __$f(x)$__ at __c__. If __$c=0$__, then the series is the __Maclaurin__ series for __f__.

$f(x) = e^{g(x)}$
$f'(x) = g'(x)e^{g(x)}$

Example 1: Find the Taylor series, centered at c, for the function.

a. $f(x) = e^{-4x}$, $c = 0$

$f(0) = 1$

$f'(x) = -4e^{-4x}$ $\quad f'(0) = -4$

$f''(x) = 16e^{-4x}$ $\quad f''(0) = 16$

$f'''(x) = -64e^{-4x}$ $\quad f'''(0) = -64$

$\vdots$ $\quad \vdots$

$f^{(n)}(x) = (-4)^n e^{-4x}$ $\quad f^{(n)}(0) = (-1)^n \cdot 4^n$

$$f(x) = e^{-4x} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{n!} \cdot x^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (4x)^n}{n!}, \quad (-\infty, \infty)}$$

$$\lim_{n\to\infty} \left| \frac{(4x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(4x)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4x\,(\cancel{n!})}{(n+1)(\cancel{n!})} \right|$$

$$= 0$$

b. $f(x)=\frac{1}{1-x}$, $c=2$

$f(x)=(1-x)^{-1}$

$f'(x)=-(1-x)^{-2}(-1)=(1-x)^{-2}$

$f''(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}$

$f'''(x)=6(1-x)^{-4}$

$f^{(4)}(x)=24(1-x)^{-5}$

$\vdots$

$f^{(n)}(x)=n!(1-x)^{-(n+1)}$

$f(2)=-1$

$f'(2)=1$

$f''(2)=-2$

$f'''(2)=6$

$f^{(4)}(2)=-24$

$\vdots$

$f^{(n)}(2)=(-1)^{n+1}n!$

$$f(x)=\frac{1}{1-x}=-1+1(x-2)-\frac{2}{2!}(x-2)^2+\frac{6}{3!}(x-2)^3-\frac{24}{4!}(x-2)^4+\dots$$

$$\boxed{f(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n,\ (1,3)}$$

Ratio test:

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(x-2)^n}\right|$$

$$=\lim_{n\to\infty}|x-2|<1$$

$$-1<x-2<1$$

$$1<x<3$$

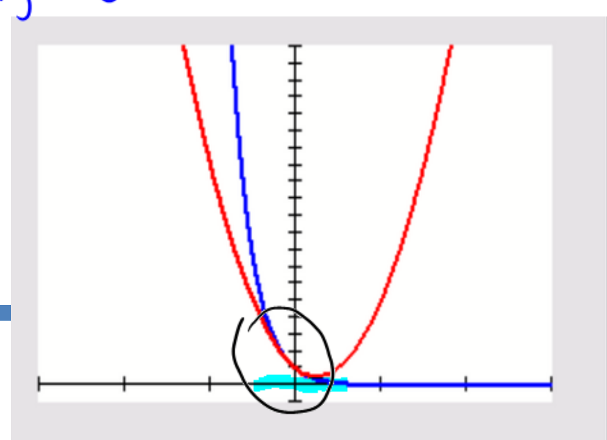
Endpoints:

$x=1$: $\sum_{n=0}^{\infty}(-1)^{n+1}(-1)^n$

$=\sum_{n=0}^{\infty}(-1)^{2n+1}$

$x=3$: $\sum_{n=0}^{\infty}(-1)^{n+1}(1)^n$

diverges by nth term test for divergence



Theorem: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Pf: For a Taylor Series, the nth Partial sum is the nth Taylor polynomial. So $S_n(x) = P_n(x)$ and

$$P_n(x) = f(x) - R_n(x)$$

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} P_n(x)$$

$$= \lim_{n\to\infty} (f(x) - R_n(x))$$

$$= f(x) - \lim_{n\to\infty} R_n(x)$$

So for any x the Taylor series converges if $\lim_{n\to\infty} R_n(x) = 0$. //

Example 2: Prove that the Maclaurin series for $f(x) = \cos x$ converges to $f(x)$ for all x.

$$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$f^{(n+1)}(x) = \pm \sin x \text{ or } \pm \cos x$$

So $\left|f^{(n+1)}(z)\right| \le 1$ for all z.

By Taylor's Thm,

$$0 \le |R_n(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}$$

and,

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

So $\lim_{n\to\infty} \left|\frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}\right| = 0$ by the squeeze theorem.

It follows that $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ converges to $f(x) = \cos x$, for all x.

Binomial Series

Let's check out the function $f(x)=(1+x)^k$, where k is a rational number. What do you think the Maclaurin series is for this function? Guess what…YOU KNOW HOW TO FIND IT!!! So, on your mark, get set, GO!

1. Differentiate $f(x)$ a bunch of times and evaluate each derivative at 0. Evil plan: Discover a pattern.

$f(x)=(1+x)^k$

$f'(x)=k\cdot(1+x)^{k-1}$

$f''(x)=k\cdot(k-1)(1+x)^{k-2}$

$f'''(x)=k(k-1)(k-2)(1+x)^{k-3}$

$\vdots$

$f^{(n)}(x)=k(k-1)(k-2)\cdots(k-(n-1))(1+x)^{k-n}$

$f^{(n)}(x)=k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$

$f(0)=1$

$f'(0)=k$

$f''(0)=k(k-1)$

$f'''(0)=k(k-1)(k-2)$

$f^{(n)}(0)=k(k-1)(k-2)\cdots(k-n+1)$

$$(1+x)^k=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots+\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$$

2. Determine the interval of convergence...Don't forget to test the endpoints!

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\cancel{k}(\cancel{k-1})(\cancel{k-2})\cdots\overbrace{(\cancel{k-n+1})}^{k-(n-1)}(k-[(n+1)-1])x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n\cdot \cancel{k}\cdot(\cancel{k-1})(\cancel{k-2})\cdots(\cancel{k-n+1})}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(k-n)\cancel{n!}\cdot x}{(n+1)\cancel{n!}}\right|$$

$= |x| < 1$

$-1 < x < 1$

Endpoints:

$x=-1$

$$\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)(-1)^n}{n!}$$ converges $-1<k<0$

and diverges for $k>0$.

3. Determine whether the series ____________ to $f(x)$ within the ____________ of ____________.

Guidelines for Finding a Power Series

1. Differentiate $f(x)$ and evaluate each derivative at c until you find a pattern.

2. Form the Taylor coefficient $a_n = \frac{f^{(n)}(c)}{n!}$, and determine the interval of convergence for the power series.

3. Determine whether the series converges to $f(x)$ within the interval of convergence.

Example 3: Find the Maclaurin series for the function using the binomial series.

a. $f(x) = \dfrac{1}{(1+x)^4} = (1+x)^{-4}$; $k = -4$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$$
$$= 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

$$(1+x)^{-4} = 1 + (-4)x + \frac{(-4)(-5)x^2}{2!} + \frac{(-4)(-5)(-6)x^3}{3!} + \frac{(-4)(-5)(-6)(-7)x^4}{4!} + \frac{(-4)(-5)(-6)(-7)(-8)x^5}{5!} + \cdots$$

$$(1+x)^{-4} = 1 - 4x + \frac{5\cdot 4x^2}{2!} - \frac{6\cdot 5\cdot 4x^3}{3!} + \frac{7\cdot 6\cdot 5\cdot 4x^4}{4!} - \frac{8\cdot 7\cdot 6\cdot 5\cdot 4x^5}{5!} + \cdots$$

$$(1+x)^{-4} = \underset{n=0}{1} - \underset{n=1}{4x} + \underset{n=2}{\frac{5!x^2}{3!2!}} - \underset{n=3}{\frac{6!x^3}{3!3!}} + \underset{n=4}{\frac{7!x^4}{3!4!}} - \underset{n=5}{\frac{8!x^5}{3!5!}} + \cdots + \frac{(-1)^n(n+3)!x^n}{3!n!} + \cdots$$

$$(1+x)^{-4} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+3)!}{3!n!}x^n$$

b. $f(x)=\sqrt{1+x^3}=(1+x^3)^{1/2}$; $k=\frac{1}{2}$

$$(1+x)^k = 1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\frac{k(k-1)(k-2)(k-3)x^4}{4!}+\cdots$$

$$(1+x)^{1/2} = 1+\frac{1}{2}x+\frac{(\frac{1}{2})(-\frac{1}{2})x^2}{2!}+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3!}+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})x^4}{4!}+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})x^5}{5!}+\cdots$$

$$(1+x)^{1/2} = 1+\frac{1}{2}x-\frac{(\frac{1}{2})^2x^2}{2!}+\frac{(\frac{1}{2})^3\cdot 3\cdot x^3}{3!}-\frac{(\frac{1}{2})^4\cdot 3\cdot 5\cdot x^4}{4!}+\frac{(\frac{1}{2})^5\cdot 3\cdot 5\cdot 7x^5}{5!}+\cdots$$

doesn't work

$$(1+x)^{1/2} = \underset{n=0}{1}+\underset{n=1}{\frac{1}{2}x}-\underset{n=2}{\frac{1\cdot x^2}{2^2\,2!}}+\underset{n=3}{\frac{1\cdot 3\cdot x^3}{2^3\,3!}}-\underset{n=4}{\frac{1\cdot 3\cdot 5\cdot x^4}{2^4\,4!}}+\underset{n=5}{\frac{1\cdot 3\cdot 5\cdot 7x^5}{2^5\,5!}}\pm\cdots$$

$$+\frac{(-1)^{n+1}\,1\cdot 3\cdot 5\cdot 7\cdots(2n-3)x^n}{2^n n!}+\cdots$$

$2n+1$
$2n-1$
$2n+3$
$2n-3$

$$(1+x)^{1/2} = 1+\frac{1}{2}x+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\cdot 1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n n!}$$ so,

$$\boxed{(1+x^3)^{1/2} = 1+\frac{1}{2}x^3+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\,1\cdot 3\cdot 5\cdots(2n-3)x^{3n}}{2^n n!}}$$

A Basic List of Power Series for Elementary Functions

FUNCTION	INTERVAL OF CONVERGENCE
$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots+(-1)^n(x-1)^n+\cdots=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$	$0<x<2$
$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots+(-1)^n x^n+\cdots$	$-1<x<1$
$\ln x = (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^{n+1}(x-1)^n}{n}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}$	$0<x\le 2$
$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x=\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$	$-1\le x\le 1$
$\arcsin x=x+\frac{x^3}{2\cdot 3\cdot 4}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\cdots+\frac{(2n)!\,x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\le x\le 1$
$(1+x)^k=1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\cdots$	$-1<x<1$*

*convergence at endpoints depends on k

Example 4: Find the Maclaurin series for the function using the basic list of power series for elementary functions.

a. $f(x) = \ln(1+x^2)$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

$$\ln(x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left[(x^2+1)-1\right]^n}{n}$$

$$\boxed{\ln(x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n}}$$

IOC = $\boxed{}$

b. $f(x) = e^x + e^{-x}$

$$(-x)^n = \left[(-1)x\right]^n = (-1)^n x^n$$

$$(ab)^n = a^n b^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$+\ e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

$$e^x + e^{-x} = (1+1) + (x-x) + \left(\frac{x^2}{2!} + \frac{x^2}{2!}\right) + \left(\frac{x^3}{3!} - \frac{x^3}{3!}\right) + \left(\frac{x^4}{4!} + \frac{x^4}{4!}\right) \dots$$

$$e^x + e^{-x} = 2 + 0 + \frac{2x^2}{2!} + 0 + \frac{2x^4}{4!} + \dots$$

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

$$\boxed{e^x + e^{-x} = 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

$$e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} x^n$$

$$= 2 + 0x + \frac{2x^2}{2!} + \dots$$

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

c. $f(x) = \cos^2 x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\left[1+\cos 2x\right]$$

$$= \frac{1}{2}\left[1+\sum_{n=0}^{\infty} \frac{(-1)^n [2x]^{2n}}{(2n)!}\right]$$

$$= \frac{1}{2}+\frac{1}{2}\cdot\left[1-\frac{4x^2}{2!}+\frac{16x^4}{4!}-\frac{64x^6}{6!}+\cdots\right]$$

$$= \frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\cdot 2^{2n}x^{2n}}{(2n)!}$$

$$\frac{2^{2n}}{2^1} = 2^{2n-1}$$

$$= \boxed{\frac{1}{2}+\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,2^{2n-1}x^{2n}}{(2n)!}}$$

d. $f(x) = x\cos x$

$$x\cos x = x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$x\cos x = x\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right]$$

$$x\cos x = \left[x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\cdots\right]$$

$$\boxed{x\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}}$$

e. $f(x) = \cot x$

Example 5: Find the first four nonzero terms of the Maclaurin series for the function $f(x)=e^x\ln(1+x)$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$e^x \ln(1+x) = \left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!}\right]$$

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right]\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right]$$

$$= x + \left(x^2 - \frac{x^2}{2}\right) + \left(-\frac{x^3}{2} + \frac{x^3}{3} + \frac{x^3}{2!}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{2!\cdot 2} + \frac{x^4}{3!}\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^4}{3} + \frac{x^4}{6}$$

$$P_4(x) = \boxed{x + \frac{x^2}{2} + \frac{x^3}{3} + 0}$$

Example 6: Use a power series to approximate the value of the integral with an error less than 0.0001.

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\int_0^{1/2} \arctan x^2 dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} dx$$

$$= \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+2)+1}}{(2n+1)(4n+2+1)} \Bigg|_{x=0}^{x=1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)} \Bigg|_{x=0}^{x=1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} \left[\left(\frac{1}{2}\right)^{4n+3} - (0)^{4n+3}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)2^{4n+3}}$$

$$\frac{(-1)^n}{(2n+1)(4n+3)2^{4n+3}} < 0.0001$$

It turns out that $n=2$ [trial & error]